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LAMINAR BOUNDARY LAYERS OF CO-CURRENT GAS-LIQUID STRATIFIED FLOWS—I. THEORY

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Abstract—The flow in the developing laminar boundary layers on a flat moving gas–liquid interface is studied theoretically. Analysis of the equations shows that flows depend on two parameters, each less than unity. Using a perturbation method, an analytical solution with accuracy to the third power of the small parameters is obtained and is in close agreement with the exact solution of the problem.

1. INTRODUCTION

Investigations of the gas–liquid flow around a moving interface are of interest to boundary layer theory in order to study heat- and mass-transfer in two-phase systems. For this purpose the equations of motion for both fluids have to be solved using the Prandtl approach. For the case of a co-current flow the equations and the boundary conditions are, figure 1:

$$\begin{aligned}&2f''' + f''f = 0,\\ &\varphi''' + 2\varphi''\varphi = 0,\\ &f(0) = \varphi(0) = 0, \quad f'(\infty) = \varphi'(\infty) = 1,\\ &f'(0) = \theta_1\varphi'(0), \quad \varphi''(0) = -2\theta_2 f''(0).\end{aligned} \qquad [1]$$

The boundary conditions express the requirements for continuity of velocities and shear stresses at the interface as well as the requirements of uniform potential flow away from the moving boundary. Here:

$$\theta_1 = \frac{U_L^\infty}{U_G^\infty}; \quad \theta_2 = \frac{\mu_G U_G^\infty}{\mu_L U_L^\infty}\sqrt{\left(\frac{U_G^\infty \nu_L}{U_L^\infty \nu_G}\right)}, \qquad [2]$$

and the functions $f(\eta)$ and $\varphi(\zeta)$ are related to the velocity components u and v, the velocities in the potential flows U_G^∞ and U_L^∞ respectively for gas and liquid, and the dynamic and kinematic viscosities μ and ν by the equations:

$$u_G = U_G^\infty f', \quad v_G = \sqrt{\left(\frac{U_G^\infty \nu_G}{4x}\right)}(\eta f' - f), \quad \eta = y\sqrt{\left(\frac{U_G^\infty}{\nu_G x}\right)}, \quad y \geq 0, \qquad [3]$$

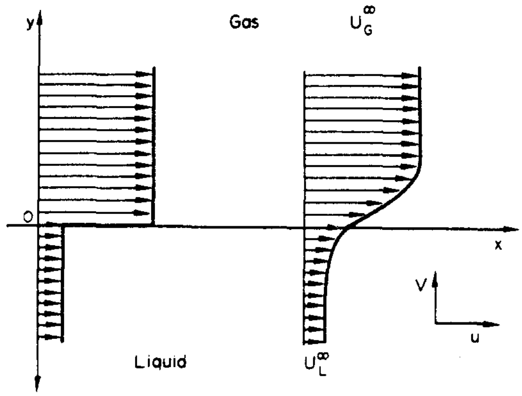

Figure 1. Sketch of the flow.

and

$$u_L = U_L^{\infty}\varphi', \quad v_L = -\sqrt{\left(\frac{U_L^{\infty}\nu_L}{x}\right)}(\zeta\varphi' - \varphi), \quad \zeta = -y\sqrt{\left(\frac{U_L^{\infty}}{4\nu_L x}\right)}, \quad y \leq 0.$$

The problems [1] were solved numerically by Keulegan (1944), Lock (1951) and Potter (1957) without any significant restrictions on the values of θ_1 and θ_2. The method of von Karman was applied to obtain initial approximations used for the numerical solution of the problems. Analytical expansion of the solution for the case when $\theta_1 \ll 1$ and $\theta_2 \ll 1$ was given by Boyadjiev (1971) using a perturbation method with an accuracy to the second power of the small parameters.

The purpose of the present work is to prove the validity of the perturbation method, in this case via comparison of the velocity profiles obtained with the results of exact solutions. The expansions proportional to the third power of the small parameters θ_1 and θ_2 are obtained. This provides a higher accuracy of the solutions and a larger interval of their practical applicability.

2. ANALYSIS

In the case when $U_G^{\infty} > U_L^{\infty}$, the parameters θ_1 and θ_2 are usually less than unity. The solution of [1] is carried out by expanding $f(\eta)$ and $\varphi(\zeta)$ in power series of θ_1 and θ_2. Thus two main questions arise:

(i) Is it admissible to expand the functions $f(\eta)$ and $\varphi(\zeta)$ as power series in θ_1 and θ_2?

(ii) How many terms in these expansions are sufficient for practical use?

An answer of the first question is given by comparison of the results obtained by the use of the approximate analytical solution to the exact numerical solution.

By [2], θ_1 and θ_2 are related as follows:

$$\theta_2\sqrt{(\theta_1^3)} = (\mu_G/\mu_L)\sqrt{(\nu_L/\nu_G)}, \tag{4}$$

i.e. intervals for θ_1 and θ_2 exist, where they both are less than unity. Let $0 < \theta_1^{\mathrm{I}} \leq \theta_1 \leq \theta_1^{\mathrm{II}} < 1$; $1 > \theta_2^{\mathrm{I}} \geq \theta_2 \geq \theta_2^{\mathrm{II}} > 0$. For rapid convergence of the analytical expansion it is not possible to choose θ_1 and θ_2 arbitrarily; the width of the intervals $[\theta_i^{\mathrm{I}}, \theta_i^{\mathrm{II}}]$ depends on the required accuracy of the solutions and the number of terms retained in the series expansions. For example, for air–water flow the error will not exceed 1% if $0.06 \leq \theta_1 \leq 0.2$ and $0.2 \geq \theta_2 \geq 0.03$, when the terms proportional to the third power of θ_i are retained. Omission of these higher order terms leads to a decrease of the interval width for 1% accuracy. For practical use it is sufficient to obtain the terms proportional to $\theta_1^m\theta_2^n$, where $m + n = 3$.

3. SOLUTIONS OF THE BOUNDARY LAYER EQUATIONS

The problem [1] is linearized after expanding $f(\eta)$ and $\varphi(\zeta)$ in the above mentioned power series. The resulting velocity profiles are:

$$f'(\eta) = f_0'(\eta) + \left(\frac{\theta_1}{\alpha} + \sqrt{\pi}\theta_1\theta_2 - \alpha\theta_1\theta_2^2\right)f_0''(\eta) + (\theta_1^2 + 2\alpha\sqrt{\pi}\theta_1\theta_2^2)f_1'(\eta) + \theta_1^3 f_2'(\eta), \tag{5}$$

$$\varphi'(\zeta) = 1 + \sqrt{\pi}(\alpha\theta_2 + \alpha_1\theta_1^2\theta_2)\,\mathrm{erfc}\,\zeta + \sqrt{\pi}\alpha^2\theta_2^2\left[-\zeta\exp(-\zeta^2)\,\mathrm{erfc}\,\zeta + \frac{\sqrt{\pi}}{2}\,\mathrm{erf}\,\zeta\cdot\mathrm{erfc}\,z - \frac{2}{\sqrt{\pi}}\exp(-\zeta^2) + \frac{1}{\sqrt{\pi}}\exp(-2\zeta^2)\right] + \theta_2^3\varphi_2'(\zeta), \tag{6}$$

where $\alpha = f_0''(0) = 0.3320$ and $\alpha_1 = f_1''(0) = -0.5447$. The unknown functions in [5] and [6] are solutions of the linear differential equations set:

$$\begin{aligned} &2f_0''' + f_0''f_0 = 0,\\ &2f_1''' + f_0f_1'' + f_0''f_1 = -\alpha^2 f_0' f_0''',\\ &2f_2''' + f_0f_2'' + f_0''f_2 = -\alpha^{-1}(f_0'''f_1 + f_0'f_1'''),\\ &f_0(0) = f_0'(0) = 0;\quad f_0'(\infty) = 1,\\ &f_1(0) = f_1'(0) = f_1'(\infty) = 0,\\ &f_2(0) = f_2'(0) = f_2'(\infty) = 0, \end{aligned} \qquad [7]$$

$$\begin{aligned} &\varphi_0''' + 2\zeta\varphi_0'' = 0,\\ &\varphi_1''' + 2\zeta\varphi_1'' = -2\varphi_0\varphi_0'',\\ &\varphi_2''' + 2\zeta\varphi_2'' = -2(\varphi_0\varphi_1'' + \varphi_0''\varphi_1),\\ &\varphi_0(0) = \varphi_0'(\infty) = 0,\quad \varphi_0''(0) = -2\alpha,\\ &\varphi_1(0) = \varphi_1'(\infty) = \varphi_1''(0) = 0\quad [\varphi_0'(0) = \alpha\sqrt{\pi}],\\ &\varphi_2(0) = \varphi_2'(\infty) = \varphi_2''(0) = 0\quad [\varphi_1'(0) = -\alpha]. \end{aligned} \qquad [8]$$

The functions f_0 and f_1 were tabulated by Boyadjiev & Piperova (1971) and f_2 is presented graphically in figure 2. The functions φ_0 and φ_1 are obtained by Boyadjiev (1971). They are the integrals of the functions in the parenthesis in [6]. The functions φ_2 could be obtained analytically by the use of Green's functions, but for practical use it is more convenient to use its graphical presentation (see figure 3).

For checking the accuracy of [5] and [6] "exact" numerical solutions of [1] were carried out. The results obtained are shown on figures 4, 5 and 6. It is found in figures 4 and 5 that the velocity results for f' and φ' obtained by means of the two methods are fully coincident in the limits of the graphical presentation. The deviation of φ' calculated from [6] in figure 6 (the dotted line) is due the large value of θ_2.

4. CONCLUSIONS

Velocity profiles in laminar boundary layers at the moving interface in a co-current gas–liquid flow are obtained using the expansion method. The good coincidence of these results with the

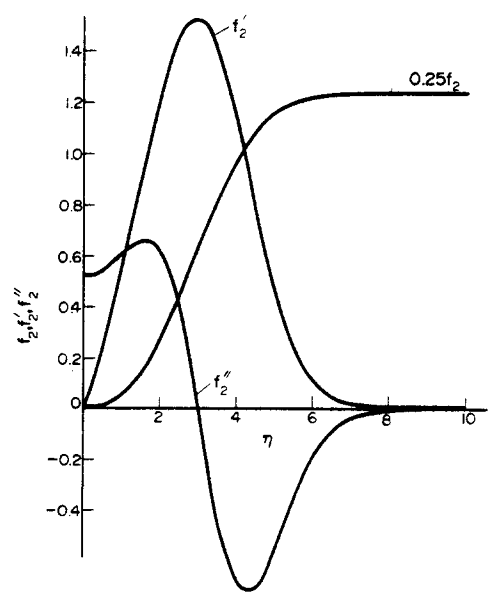

Figure 2. The function $f_2(\eta)$ and its derivatives.

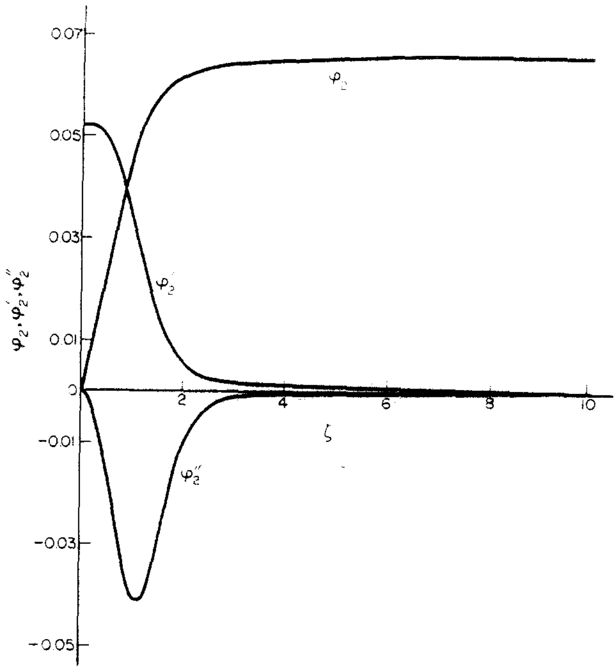

Figure 3. The function $\varphi_2(\zeta)$ and its derivatives.

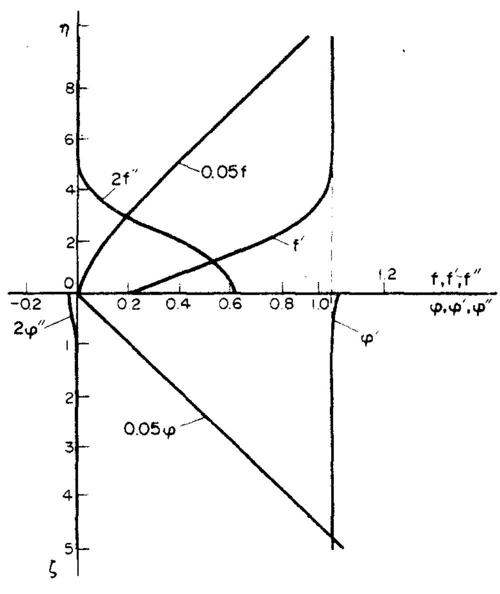

Figure 4. The functions $f'(\eta)$ and $\varphi'(\zeta)$ obtained by the "exact" numerical solution of [1] and by the perturbation method ([5] and [6]) for $\theta_1 = 0.200$; $\theta_2 = 0.054$ ($f'(0) = 0.206$; $\varphi'(0) = 1.031$). The functions $f(\eta)$, $f''(\eta)$, $\varphi(\zeta)$ and $\varphi''(\zeta)$—obtained by the "exact" solution of [1] ($f''(0) = 0.311$; $\varphi''(0) = -0.016$).

ones derived from the "exact" numerical solutions leads to the following conclusions:

(i) The functions $f(\eta)$ and $\varphi(\zeta)$ can be expanded in series of the powers of θ_1 and θ_2.

(ii) The equations obtained for $f(\eta)$ and $\varphi(\zeta)$ are quite accurate for practical use, when the third powers of θ_1 and θ_2 are retained.

(iii) The method of perturbation can be successfully used for calculation of the subsequent approximations in principle without any computational difficulties.

(ix) Increase of the accuracy of [5] by decreasing θ_1 leads to decrease of the accuracy of [6] and conversely.

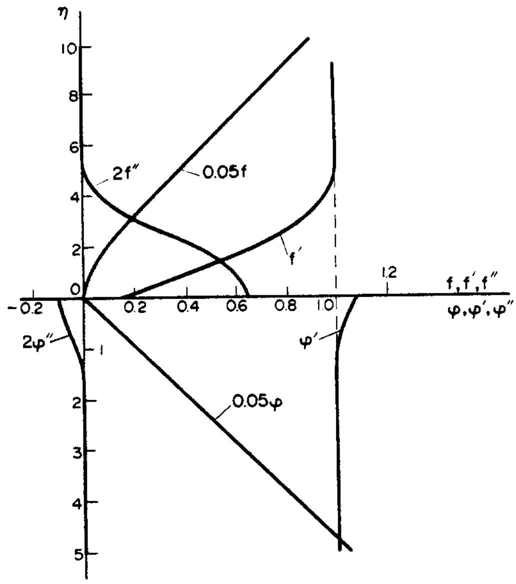

Figure 5. The functions $f'(\eta)$ and $\varphi'(\zeta)$ obtained by the "exact" numerical solution of [1] and by the perturbation method ([5] and [6]) for $\theta_1 = 0.100$; $\theta_2 = 0.152$ ($f'(0) = 0.110$; $\varphi'(0) = 1.087$). The functions $f(\eta)$, $f''(\eta)$, $\varphi(\zeta)$ and $\varphi''(\zeta)$—obtained by the "exact" solution of [1] ($f''(0) = 0.327$; $\varphi''(0) = -0.049$).

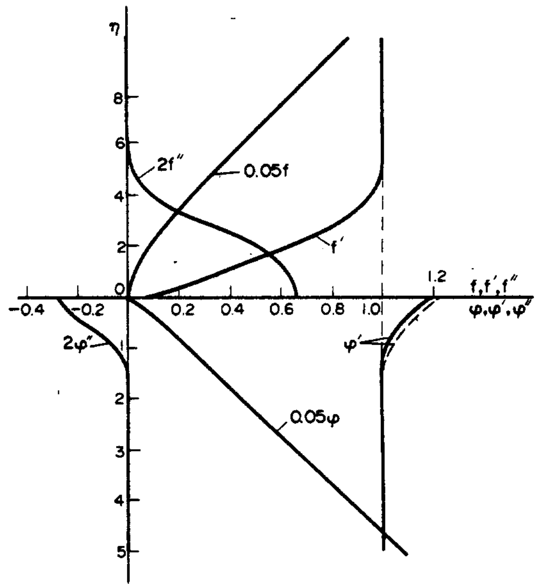

Figure 6. The functions $f'(\eta)$ and $\varphi'(\zeta)$ obtained by the "exact" numerical solution of [1] and by the perturbation method ([5] and [6]) for $\theta_1 = 0.050$; $\theta_2 = 0.600$ ($f'(0) = 0.603$; $\varphi'(0) = 1.232$). The functions $f(\eta)$, $f''(\eta)$, $\varphi(\zeta)$ and $\varphi''(\zeta)$—obtained by the "exact" solution of [1] ($f''(0) = 0.330$; $\varphi''(0) = -0.142$).

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